

Dirichlet Joyce Manifolds, Discrete Torsion and Duality

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Abstract

Using U -duality transformations we map perturbative Type IIA string theory compactified on a class of Joyce 7-manifolds to a D -strings on D -manifold description in Type IIB theory. For perturbative Type IIB theory on the same class of Joyce manifolds we use duality transformations to map to an M -theory, M -manifold description, which is an orientifold with fivebrane twisted sectors. D and M -manifold analogues of Joyce orbifolds with discrete torsion are found. For the same class of compactifications we show that Type IIA/IIB theory on a Joyce orbifold without (with) discrete torsion is T -dual to Type IIB/IIA theory on the same orbifold with (without) discrete torsion. For this class of Type II compactifications this proves an extension of the Papadopoulos-Townsend conjecture, which states that the Type IIA and IIB theories compactified on the same Joyce 7-manifold are equivalent. Finally we note that the Papadopoulos-Townsend conjecture is a special case of the Generalised Mirror Conjecture.

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1 Introduction

U -duality, [1], interchanges perturbative string states (for example standard twisted sector states of an orbifold compactification) with D -branes [2] and M -branes [3]. For example, in the case of Type IIA on $K3$, U -duality transforms the “standard” perturbative description into a Type IIB, D -strings on D -manifold description [5, 6, 4]. For the simple case when $K3$ is the Z_2 orbifold of T^4 , U -duality exchanges the twisted sector IIA states with configurations of RR D -fivebranes in the IIB theory [5]. For the case of Type IIB on T^4/Z_2 , the perturbative compactification can be transformed to an orientifold of M -theory [7, 8], in which M -fivebranes carry the duals of the twisted sector states [8]².

In this sense, D -brane and M -brane configurations encode information about U -dual compactification spaces³. The relationship between branes, geometry and physics has been investigated in a variety of fascinating contexts in recent months (see for example [9]).

The purpose of this note is twofold. The first is to exploit the analogy between the construction of $K3$ as a blown up toroidal orbifold and the construction of manifolds of exceptional holonomy using the same technique [10, 11, 12]. This will give us a simple generalisation of the D/M -manifold descriptions of Type IIA/IIB on T^4/Z_2 , to the case of Joyce 7-manifolds of G_2 holonomy. In doing this we will also find further examples of D and M -manifold descriptions of discrete torsion, as have been found recently in [13, 14]. We find that the geometry of the Joyce manifold compactification is, as expected, beautifully encapsulated by the fivebrane twisted sectors. We present these constructions in sections two and three. These results are evidence that Joyce 7-manifolds have points in their moduli space in which A_n singularities appear.

The second purpose of this note is to explore what may loosely be called “mirror conjectures” for Type II compactifications on Joyce 7-manifolds. In [15], the conformal field theory description of spaces of exceptional holonomy was given. In order to interpret certain results of that description, Shatashvili and Vafa proposed a Generalised Mirror Conjecture, which is

²For convenience, we will refer to such backgrounds of M -theory as M -manifolds.

³For instance, couplings in a compactified theory are determined by topological intersection numbers of the compactifying space. In principle, these should be determinable in the dual D or M -manifold description.

proposed to apply to any quantum sigma model. Essentially this conjecture states that if any ambiguity arises in the determination of topological properties of the target space using quantum field theory, then there exists a dual quantum field theory which resolves the ambiguity. A classic example is given by the Witten index, $Tr(-)^F$. In 2d supersymmetric sigma models, this index computes the Euler character of the target space, up to an overall sign ambiguity [17]. Thus, in this case, the Shatashvili-Vafa (SV) conjecture requires the existence of two dual sigma models, which differ by a sign in the calculation of the Witten index on supersymmetric ground states.

On the other hand, compactification of both the Type IIA and IIB theories on a general 7-manifold of G_2 holonomy, J , was considered in [18]. There it was observed that the massless spectra of the two 3d theories agreed (after dualising all vectors to scalars), and it was conjectured on that basis that both the IIA and IIB theories compactified on the same G_2 7-manifold are physically equivalent, or dual. We will refer to this conjecture as the PT conjecture. The PT conjecture, as stated in [18], applies to an arbitrary manifold of G_2 holonomy. In this sense, it applies to IIA/IIB backgrounds which are well defined in classical geometry. However, because such backgrounds admit a description as conformal field theories, it is natural to propose that the PT conjecture applies to “ G_2 ” backgrounds which do not have an interpretation in classical geometry, but are well defined in CFT.

However, in this paper, we will restrict our attention to the largest class of manifolds constructed in [11]. Specifically we will consider manifolds of G_2 holonomy constructed as desingularisations of T^7/Z_2^n . In principle, there may exist orbifolds in this class which admit no desingularisation in classical geometry and our analysis applies to these cases as well; but as examples we will only consider cases considered by Joyce [11].

For the T^7/Z_2^n orbifolds, it turns out (following [19]) that we will be able to prove an extension of the PT conjecture. We will show that the Type IIA/IIB theory on a T^7/Z_2^n orbifold without (or with) discrete torsion transforms under a unique T -duality transformation to the IIB/IIA theory on the same orbifold with (or without) discrete torsion. However, the result of [15] states that the IIA or IIB theory on such an orbifold without discrete torsion is equivalent to the same theory on the orbifold with discrete torsion, up to deformations in the moduli space. Thus, given one of the Type II theories on the orbifold without discrete torsion, one can marginally perturb the theory and one finds the same theory on the same orbifold, but now with

discrete torsion turned on.

The results we find here are therefore wholly analagous to the results concerning mirror symmetry for Calabi-Yau $T^6/[Z_2 \times Z_2]$ orbifolds [19]. It is in this sense that PT conjecture as stated in [18] is extended for the orbifold cases we consider here to include discrete torsion. One finds that the dual theory is in fact a marginal perturbation of the theory one expected from the conjecture in [18].

The above comments also beg the question: Is there any relationship between the PT conjecture and the SV conjecture? For the class of Joyce compactifications that we consider here, we provide an answer to this question. It turns out that the SV and PT conjectures are in fact the same. Thus the T -duality between IIA and IIB string theories on T^7/Z_2^n orbifolds with and without discrete torsion give concrete examples of cases for which the SV generalised mirror conjecture is satisfied.

2 Dirichlet Joyce Manifolds.

In [11] Joyce constructed many examples of compact 7-manifolds with G_2 holonomy as blown up orbifolds of the seven torus, T^7 . The largest class of such manifolds presented in [11] were constructed with a Z_2^n orbifold group. Of these n Z_2 generators, three act non-freely on T^7 (each non-free Z_2 inverting four coordinates). If one compactifies a higher dimensional locally supersymmetric theory on such a Joyce space then each non-free Z_2 breaks half of the N supersymmetries which are intact after compactification on T^7 , leaving $N/8$. The remaining $n-3$ Z_2 generators preserve supersymmetry and act freely on the torus. We will mainly consider $T^7/(Z_2^3)$ Joyce orbifolds in this paper, but in principle, our analysis also applies to the cases with the additional freely acting generators.

Begin with Type IIA string theory on a T^7 with coordinates x_i , for $i = 1, 2, \dots, 7$. This theory has $N = 16$ supersymmetry in three dimensions. In accord with our above comments, we can take a Z_2^3 orbifold of this theory which breaks supersymmetry to $N = 2$. The three Z_2 generators of the orbifold group, Γ may be defined as follows [11]:

$$\alpha(x_i) = (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7) \quad (1)$$

$$\beta(x_i) = (b_1 - x_1, b_2 - x_2, x_3, x_4, -x_5, -x_6, x_7) \quad (2)$$

$$\gamma(x_i) = (c_1 - x_1, x_2, c_3 - x_3, x_4, -x_5, x_6, -x_7) \quad (3)$$

The constants, b_i and c_j take values in $(0, 1/2)$, and remain to be specified. Of course the *most* general such orbifold can have translations in any of the inverted directions for each generator. This requires specifying four constants for each generator, and there exist 16 possible choices for each generator. This gives 16^3 different orbifolds that one can define. However it is very likely that there exist large degeneracies between such orbifolds, and the number of independent string backgrounds that one can obtain is probably much smaller. The analysis of the present paper in principle applies to all of these cases, but for simplicity we restrict ourselves to the class of orbifolds defined above.

Note that there are no shifts in the x_4, x_5, x_6, x_7 directions. Desingularisation of the T^7/Γ orbifold leads to a compact 7-manifold with holonomy precisely G_2 , as long as $(b_1, b_2) \neq (0, 0)$ and $(c_1, c_3) \neq (0, 0)$ ⁴. In this paper, we restrict ourselves to a study of the cases for which classical desingularisations are known, although the analysis will also apply to the other cases.

For notational simplicity, we will denote an element of the orbifold group which inverts for example the x_l, x_m, x_n, x_p coordinates as $I_{lmnp}(a_1, a_2, a_3)$, where the constants a_1, a_2, a_3 specify shifts in the x_1, x_2, x_3 directions respectively. So for example, Type IIA theory on the Z_2 orbifold defined by β will be denoted by Type IIA on $T^7/I_{1256}(b_1, b_2, 0)$, and so on.

Thus Type IIA string theory on the Joyce orbifold defined above will be denoted as
Type IIA on

$$T^7/[I_{1234}(0, 0, 0), I_{1256}(b_1, b_2, 0), I_{1357}(c_1, 0, c_3)] \quad (4)$$

The non-zero Betti numbers of a compact 7-manifold of G_2 holonomy are $b_0=b_7=1$, $b_2=b_5$ and $b_3=b_4$ (These should not be confused with the constants b_i above.). The conformal field theory description of such string backgrounds

⁴For the cases when the constants do not satisfy both of these constraints, it is at present unknown how to desingularise the orbifold, although it is still plausible that such a desingularisation does exist [20]. In string theory however there does not seem to be any obstruction to the definition of the orbifold for any choice of the constants and one can calculate topological properties of the compactification space in the standard way. If it does indeed turn out that such singularities cannot be resolved using classical geometry, then these cases will be further examples of stringy geometry.

was given in [15](see also [16]). M -theory and string theory compactification on such spaces were first studied in [18] in the context of duality. There it was shown that, after dualising all vectors to scalars, the perturbative massless spectrum of both the Type IIA and IIB string theories on such Joyce manifolds consisted of $d = 3$, $N = 2$ supergravity with $b_2 + b_3 + 1$ scalar multiplets. Before dualising the vectors, however, the spectrum in the Type IIA theory is $b_2 + 1$ vector multiplets and b_3 scalar multiplets. In the Z_2^n orbifolds, b_2 vector multiplets and $b_3 - 7$ scalar multiplets arise in the twisted sectors, with one vector and 7 scalar multiplets in the untwisted sector. Here we will be interested in the U -dual description of such compactifications. This description will take the form of D -manifolds and M -manifolds.

Consider first setting the constants $(b_1, b_2, c_1, c_3) = (0, 1/2, 1/2, 1/2)$. In [11] it was shown that desingularisation of the orbifold defined by equations (1) – (3) with this choice of constants gives a Joyce manifold with $b_2 = 12$ and $b_3 = 43$. The Type IIA orbifold description is denoted by: Type IIA on

$$T^7/[I_{1234}(000), I_{1256}(0, 1/2, 0), I_{1357}(1/2, 1/2, 0)] \quad (5)$$

Now make an $R \rightarrow 1/R$ T-duality transformation on the three circles in the 4, 6, 7 directions. This transformation will be denoted by T_{467} . This takes us to an orbifold of Type IIB theory denoted by: Type IIB on

$$T'^7/[I_{1234}(0, 0, 0).(-)^{F_l}, I_{1256}(0, 1/2, 0).(-)^{F_l}, I_{1357}(1/2, 1/2, 0).(-)^{F_l}] \quad (6)$$

Here, the prime indicates that this T^7 is T -dual to the original one. However, from now on (until section 4), we will drop the primes for notational simplicity, with the relationships between dual circles understood. The important point to note is that the only non-freely acting members of the orbifold group are the generators, and that in the IIB description these are all three of the form $I_{lmno}.(-)^{F_l}$, for some $l \neq m \neq n \neq o$. It was pointed out in [5], that the “twisted” sector states of such an element in the Type IIB theory consist of sixteen $NS - NS$ fivebranes⁵. In our case, this means we have three sets of such fivebranes, which in addition wrap around three sets of three-tori, giving membranes in three dimensions. Each of these membranes carries one vector multiplet and three scalar multiplets of the $N = 2$ supersymmetry. However,

⁵By “twisted” sector we will take to mean any states which are required for physical consistency of the theory.

for each set of fivebranes, we must consider the action of the orbifold group, which in this case identifies each set in four groups of four, leaving four independent fivebranes associated with each of the three generators. This gives a total of 12 vector multiplets and 36 scalar multiplets coming from the twisted sectors of the orbifold. From the untwisted sector, we find 1 vector and 7 additional scalar multiplets giving a total of 13 vector and 43 scalar multiplets. This is precisely what we found in the original perturbative Type IIA orbifold. Note that there is a perfect correspondence between untwisted and twisted sector states in both descriptions.

We can now use the Type IIB S-duality [1] Z_2 element, which exchanges all objects carrying $NS - NS$ charges with those carrying RR charges⁶. This means we have a D -strings on D -manifold description [6] of the original Type IIA compactification. The two descriptions are related by U -duality. In particular, the blowing up modes which desingularise the Joyce orbifold are mapped into wrapped RR fivebrane states in the D -manifold description. The “addition” to the (co)homology of the Joyce manifold which arises from desingularisation in the IIA description is beautifully encapsulated by the wrapped fivebrane worldvolume fields in the D -manifold description.

Let us consider a slightly more complicated example. This will involve discrete torsion. Set $(b_1, b_2, c_1, c_3) = (0, 1/2, 1/2, 0)$. Begin with Type IIA on the Joyce orbifold defined by equation (4), with this choice of shift vectors. In the untwisted sector we find 1 vector multiplet and 7 scalar multiplets. The two sets of sixteen 3-tori fixed by $I_{1234}(0, 0, 0)$ and $I_{1256}(0, 1/2, 0)$ respectively are both identified in four groups of four by the orbifold group, giving a total of eight independent contributions to the singular set of the orbifold, which are locally of the form:

$$R^4/Z_2 \times T^3 \tag{7}$$

where the Z_2 reflects all four coordinates of R^4 . Desingularising each of these contributes 1 to b_2 and 3 to b_3 of the manifold [11]. Desingularisation of the third non-freely acting element, $I_{1357}(1/2, 0, 0)$, is more complicated and involves discrete torsion [15]. This is due to the following. The element $I_{1234}(0, 0, 0).I_{1256}(0, 1/2, 0) = I_{3456}(0, 1/2, 0)$ acts trivially on the set of sixteen fixed 3-tori of $I_{1357}(1/2, 0, 0)$. The element $I_{1256}(0, 1/2, 0)$ exchanges these sixteen elements in eight pairs leaving eight singularities associated

⁶It also transforms $(-)^{F_L}$ into Ω .

with $I_{1357}(1/2, 0, 0)$, each of which is locally of the form:

$$(R^4/Z_2 \times T^3)/Z_2' \quad (8)$$

Here, Z_2' denotes the action of $I_{3456}(0, 1/2, 0)$. It turns out, that such a singularity admits two topologically distinct resolutions, the details of which may be found in [11]. This is discrete torsion. One choice of resolution contributes 1 to b_2 and 1 to b_3 . The other choice contributes 2 to b_3 only. From the Type IIA orbifold point of view, these two choices of resolution correspond to a choice in an overall phase factor acting on the twisted sector states [15]. A detailed study of orbifolds with discrete torsion was made in [19].

Of the eight singularities which admit two resolutions, if we choose l to admit the first resolution and $8 - l$ to admit the second, then all in all the non-trivial Betti numbers of the Joyce manifold are $b_2 = 8 + l$ and $b_3 = 47 - l$ [11]. This means, that in the Type IIA orbifold compactification we will find 1 vector multiplet and 7 scalar multiplets from the untwisted sector plus $8 + l$ vector multiplets and $40 - l$ scalar multiplets from the twisted sectors.

As we did in the last example, let us map this Type IIA orbifold to a Type IIB D -manifold. This can be achieved using exactly the same duality transformations as we considered before. After applying these we find a Type IIB compactification denoted by: Type IIB on

$$T^7/[I_{1234}(0, 0, 0).\Omega, I_{1256}(0, 1/2, 0).\Omega, I_{1357}(1/2, 0, 0).\Omega] \quad (9)$$

Again, the “twisted” sectors associated with each of the non freely acting elements are 16 RR fivebranes. The first two generators both contribute, as before, four membranes each.

However, when the fivebranes associated with the third generator are at its respective fixed points, the element $I_{1234}(0, 0, 0).\Omega.I_{1256}(0, 1/2, 0).\Omega = I_{3456}(0, 1/2, 0)$ preserves these fixed points. Therefore, as in [14], the action of this group element on these fivebranes (there are eight independent fivebranes associated with the third generator) must be taken into account. These eight fivebranes are the U -duals of the eight twisted sector multiplets in the discrete torsion sector of the perturbative Type IIA orbifold. The four scalar fields on the fivebranes which interest us here represent the location of the fivebrane in the (x_1, x_3, x_5, x_7) directions. The action of $I_{3456}(0, 1/2, 0)$ inverts two of these coordinates, (x_3, x_5) . We therefore expect that two of the scalars to be

odd under this action. These fivebranes are wrapped around three-tori with coordinates (x_2, x_4, x_6) , and $I_{3456}(0, 1/2, 0)$ inverts two of these, (x_4, x_6) as well as shifting x_2 by $1/2$. We therefore expect that two degrees of freedom of the fivebrane vector field will be projected out. This choice of Z_2 action means that each of these eight fivebranes will contribute 1 vector multiplet and 1 scalar multiplet to the $3d$ $N = 2$ theory. These multiplets correspond to the first choice of resolution (which added 1 to b_2 and b_3) of the singularity in equation (8) in the Type IIA dual.

However, in the Type IIA case we noted that there existed an overall phase choice of ± 1 corresponding to discrete torsion. Under the U -duality element which maps the Type IIA description to the D -manifold configuration, a consistent picture implies that a second choice of Z_2 action on these fivebrane fields is also possible here. Consequently, this means inserting an overall minus sign to the action of the Z_2 , which means that the fivebrane fields which were odd under our first choice are now even, and vice-versa. With this latter choice of Z_2 action, each of these eight fivebranes will contribute 2 scalar multiplets to the $3d$ theory. We will assume that the ambiguity in the choice of Z_2 actions is consistent⁷. With this assumption we can choose l of these fivebranes to be of the first type and $8 - l$ to be of the latter type. If we now collect together all contributions to the massless field content we find a total of $9 + l$ vector multiplets and $47 - l$ scalar multiplets in the three dimensional theory. This is precisely what we found in the perturbative Type IIA description of the theory.

Finally, it is straightforward to see that when configurations of the above D -branes coincide, one can find points in the moduli space with $U(n)$ gauge symmetry [26]. From the original perturbative Type IIA orbifold point of view this should presumably be interpreted as an appropriate D -brane wrapping around a vanishing supersymmetric cycle. In fact, since manifolds of G_2 holonomy have only 3-cycles and 4-cycles which are supersymmetric (see for instance the sixth reference in [9]). this implies that the 4-brane in Type IIA is wrapping around a vanishing 4-cycle.

⁷Presumably, this can be shown using standard D -brane techniques.

3 M -manifold Description.

In this section, we will study Type IIB theory on the Joyce 7-orbifolds defined in equations (1)-(3). We will show that this is related to a certain M -manifold configuration of fivebranes on an orientifold of M -theory. This construction is the generalisation to Joyce manifolds of the M -manifold description of Type IIB on $K3$ [8]. (The relationship between Type IIB on $K3$ and M -theory was also discussed in [7]).

We begin then with Type IIB on a Joyce 7-orbifold. This is defined as: Type IIB on:

$$T^7/[I_{1234}(0, 0, 0), I_{1256}(b_1, b_2, 0), I_{1357}(c_1, 0, c_3)] \quad (10)$$

As we mentioned earlier, in [18] it was shown that IIB theory compactified on a Joyce 7-manifold gives, after dualising all vectors to scalars, b_2+b_3+1 scalar $N = 2$ multiplets in three dimensions. However, before dualising the vectors, we find b_2+1 scalar multiplets and b_3 vector multiplets. The vector fields in the vector multiplets, come from expanding the ten-dimensional four-form potential of the IIB theory in a basis of harmonic three-forms on the manifold. (The dual of this statement is the expansion of the four-form in terms of harmonic four-forms on the manifold, giving $b_4=b_3$ scalar multiplets.) If we contrast this with the Type IIA case, we see that the number of vector multiplets in one theory is the same as the number of scalar multiplets of the other theory, and vice-versa. This is strikingly similar to mirror symmetry for Calabi-Yau threefolds. On this basis it would at first sight appear natural to conjecture a “mirror” symmetry for manifolds of G_2 holonomy, under which $b_2 \leftrightarrow b_3$, and Type IIA is interchanged with Type IIB. However, in three dimensions there is the added twist that vectors are dual to scalars, and that by dualising all vectors, the number of scalar multiplets in both the IIA and IIB theories is the same, and therefore such a conjectured symmetry may not be so straightforward. Because of this it is still natural to interchange vector multiplets with scalar multiplets, but with the possibility that b_2 and b_3 remain invariant ie the manifold is the same on both sides of the duality map. Our results in the next section will strongly favour against a symmetry which interchanges b_2 with b_3 , and as we mentioned, will prove the PT and SV conjectures, which in these cases essentially leave the Betti numbers inert⁸. A more detailed analysis, in particular quantum,

⁸ The reader is referred to the last section of this paper for the precise meaning of this

non-perturbative aspects of the duality between IIA and IIB theory on the same manifold of G_2 holonomy will appear elsewhere [21].

Again, the important point to note about the above orbifold of Type IIB is that the only non-freely acting members of the orbifold group are the generators, and that these all three invert four coordinates of the torus. It then follows from the work of [8, 24], that the above orbifold of Type IIB theory on T^7 has a description in M -theory on T^8 . We can see this more explicitly as follows. Make the T -duality transformation T_{467} on the Type IIB orbifold above. This maps us to Type IIA on

$$T^7/[I_{1234}(0,0,0).(-)^{F_l}, I_{1256}(b_1, b_2, 0).(-)^{F_l}, I_{1357}(c_1, 0, c_3).(-)^{F_l}] \quad (11)$$

Now consider M -theory on $T^8 = T^7 \times S^1$, where this T^7 is the same as that in the orbifold in equation (11) and the coordinate labelling S^1 is x_8 . Then, because of the relationship between Type IIA and M -theory [23] and a result given in [24], the above orbifold ((11)) of the Type IIA theory (and therefore by T -duality, the Type IIB Joyce orbifold ((10)), is equivalent to the following orientifold of M -theory: M -theory on

$$T^7 \times S^1/[I_{12348}(0,0,0), I_{12568}(b_1, b_2, 0), I_{13578}(c_1, 0, c_3)] \quad (12)$$

The non-freely acting elements are the generators, and all three are of the form considered in [8]. The twisted sector associated with each generator consists therefore of sixteen M -fivebranes⁹, which wrap around three-tori. Again, one must project onto states invariant under the orbifold group. For the examples of Joyce manifolds we considered in the Type IIA case, it is straightforward to check that the massless spectra in the corresponding M -manifold description that we give here are precisely what we expected from its dual description as a perturbative Type IIB compactification on a Joyce 7-manifold, namely b_2+1 scalar multiplets and b_3 vector multiplets¹⁰. We do not repeat this here for brevity, but this can be calculated along similar lines as we presented in the D -manifold description of Type IIA on a Joyce orbifold. This gives the M -manifold description of Type IIB theory on these

statement concerning the Betti numbers.

⁹It is natural to impose that the configuration of fivebranes associated with each Z_2 is that proposed in [8] in order to achieve anomaly cancellation

¹⁰In the example with discrete torsion, one must again make the additional assumption (as in [14]), that there are two consistent choices for the appropriate Z_2 action.

Joyce orbifolds. Shrinking the circle in the M -theory compactification, gives the weak coupling limit of the Type IIB compactification, as in the $K3$ case [8].

The fact that the massless spectrum is the same, after appropriate duality transformations, as that found in the Type IIA compactification does not come as a surprise, because it was pointed out in [18] that both the Type IIA and IIB theories compactified on the same Joyce 7-manifold give the same massless spectrum.

Finally, as we discussed at the end of the last section, we can make some general comments about special points in the moduli space of the above compactifications. Firstly, it was shown in [8], that when two M -fivebranes coalesce one gets a non-critical string theory, which compactified on S^1 gives a massless $SU(2)$ vector multiplet of (the equivalent of) $N = 4$ supersymmetry in four dimensions. With more complicated configurations of fivebranes, this statement has generalisations to include $A - D - E$ groups [8]. Moreover, the string theory one gets is equivalent to that obtained by wrapping a D -threebrane of IIB theory on a vanishing S^2 [27, 8]. It is clear from these comments that in the M -manifolds above, that at special points in the moduli space when fivebranes coincide, one will find precisely the string theory discussed in [8]. However, this theory will be compactified on T^3 , giving an enhancement of gauge symmetry in the full three dimensional theory.

From the original perturbative IIB orbifold point of view, this enhancement of gauge symmetry can come *only* from a threebrane wrapped around an $S^2 \times S^1$ submanifold of $S^2 \times T^3$, where the S^2 is shrinking to zero size. The number and singularity structure of such vanishing 3-cycles which arise from desingularising the orbifold is in one-to-one correspondence with similarly obtained vanishing 4-cycles in the Type IIA case of the previous section. This is in accord with the PT conjecture.

4 The PT and SV Conjectures.

In [18], it was conjectured that the Type IIA and Type IIB theories on the *same* Joyce 7-manifold are physically equivalent. We will show in this section that this is indeed true up to moduli deformations. In fact, the two are related by T -duality, as one might have anticipated.

In section two, we described the D -manifold description of Type IIA

theory on a class of Joyce 7-manifolds. This was essentially a generalisation of the D -manifold description of Type IIA on $K3$ [5, 4, 6]. In the last section, we described the M -manifold description of Type IIB on the same class of Joyce manifolds. This description was essentially a generalisation of Witten's M -manifold construction [8], to the case of G_2 holonomy. The fact that all these descriptions give the same low energy spectrum is further evidence that the conjecture of [18] is correct.

In the Type IIA compactification on the Joyce orbifolds, J , that we have considered here, we gave three different U -dual descriptions: (i) : Type IIA on J . (ii) : M -theory on $J \times S^1$. (iii) : Type IIB D -manifold.

For Type IIB on J we have: (i) : Type IIB on J . (ii) : M -theory M -manifold.

In order to show that all five of these descriptions are equivalent under duality transformations between expectation values of some of the moduli fields, it is sufficient to show that any of the first three are dual to any of the second two; the further equivalences should follow from the duality transformations we have used in previous sections. We will show, that for the class of Joyce manifolds that we have studied in this paper, Type IIA on J is T -dual to Type IIB on J' , where the difference between J and J' is given by $R \rightarrow 1/R$ transformations on some of the circles in the original T^7 ; a further difference will be that if the theory on J has no discrete torsion, then J' will have discrete torsion, and vice-versa.

Consider our original Type IIA compactification, equation (4). If we make the T -duality transformation T_{235} on this compactification we end up with the following: Type IIB on

$$T'^7/[I_{1234}(0, 0, 0), I_{1256}(b_1, b_2, 0), I_{1357}(c_1, 0, c_3)] \quad (13)$$

Here, we have reinstated the prime to indicate the T -duality map between the compactification moduli in the two theories. At first sight this, by definition, is the Type IIB theory on the (T -dual of the) Joyce manifold we started with. We would have therefore shown, that for this class of Joyce manifolds, T -duality interchanges the Type IIA and Type IIB theories compactified on the same manifold. For this class of Joyce 7-manifolds, this would give a simple proof of the PT conjecture [18]. However, there is a subtlety involved here. Orbifold conformal field theories are not unique [25]. In general it is possible to introduce discrete phases in the path integral twisted sectors under the

constraints of modular invariance [25]. In the above T -duality transformation between the two theories, additional phases in the path integral of these theories will not be transparent. These phases, if present, correspond to discrete torsion [25].

The relationship between T -duality, mirror symmetry and orbifolds with discrete torsion was investigated in detail in [19]. There it was found that for the Type IIA/B theory compactified on a Calabi-Yau $T^6/[Z_2 \times Z_2]$ orbifold without discrete torsion, the mirror theory is a T -dual compactification of the Type IIB/A theory on the orbifold with discrete torsion, and vice versa. This equivalence was proved to genus g in [19]. Given that our orbifolds here contain just an extra Z_2 generator, one would expect some analogous T -dual relationship here. The only other difference technically, between the orbifolds here and that considered in [19] is that our orbifold generators induce extra translations of the coordinates of the torus. Consider, for example, the following Calabi-Yau compactification: Type IIA/IIB on

$$T^6/[I_{1234}(0,0,0), I_{1256}(1/2,0,0)] \quad (14)$$

The only difference between this case and that considered in [19] is the presence of a non-zero translation of the coordinate in the x_1 direction. The only non-freely acting elements of the orbifold group are the two generators. The singular set of the first generator is sixteen copies of T^2 . However, these are interchanged in eight pairs by the second generator, leaving eight invariant T^2 's coming from this element. The same is true for the second generator. Its sixteen fixed tori are reduced to eight by the action of the other Z_2 . Thus altogether, we have sixteen T^2 's even under the action of $Z_2 \times Z_2$ and sixteen which are odd. (h_{11}, h_{21}) in the untwisted sector are $(3, 3)$. Each even T^2 adds one to both of these, giving $(19, 19)$ for the final Hodge numbers.

Now, let us turn on discrete torsion. This introduces an overall phase factor of -1 in the twisted sectors, and consequently means that states which were even(odd) are now odd(even). In this case, because the states which were odd without discrete torsion also constitute sixteen tori, the end result is to produce a conformal field theory which also computes $(19, 19)$ for the Hodge numbers. This is of course expected as $h_{11} = h_{21}$ for the original orbifold. The result of [19] also applies to this case, and the case without discrete torsion is T -dual to the case with discrete torsion, and vice versa. The requisite transformation in this case is T_{246} . Further, the T -duality transformation is the mirror transformation in this case also.

Thus, formally we may conclude that the Type IIA/IIB compactification (14) without discrete torsion is mirror to the IIB/IIA compactification (14) with discrete torsion. We now move on to discuss the compactifications which have been the focus of this paper.

Above we showed that, under the transformation T_{235} , the orbifold generators of our original Type IIA compactification (without discrete torsion), transform into the same generators of a Type IIB compactification. We have not yet checked whether or not this IIB compactification has discrete torsion turned on. All that is required now is a straightforward generalisation of the computation in [19], of how the path integral for the theory transforms under T_{235} . As pointed out in [19], this can be done by computing the transformation of the path integral measure. For definiteness we will restrict ourselves to our example orbifold which had $(b_1, b_2, c_1, c_3) = (0, 1/2, 1/2, 0)$, but the result is independent of the shift vectors in equations (2) - (3).

Consider again then this Type IIA orbifold *without* discrete torsion. (The Betti numbers for the classical manifold in this case are $b_2=8+l$ and $b_3=47-l$, with $l = 0, 1, \dots, 8$). The orbifold isometry group is $(Z_2)^3$, so the discrete torsion group is $(Z_2)^2$ [25], as for each element, there is a $Z_2 \times Z_2$ choice for the overall phase in the path integral associated with the other generators of the group. As we discussed earlier using fivebrane twisted sectors, in this example (without discrete torsion), the singular set associated with each of three non-freely acting elements is sixteen three-tori, each of which get interchanged by the remaining $Z_2 \times Z_2$ group. The first and second generators contribute four independent three-tori each. However as we discussed, the third generator contributes a further eight three-tori which are however acted on by a further Z_2 (see equation (8)). All in all this means that the Betti numbers of the orbifold with no discrete torsion are $b_2 = 16$ and $b_3 = 39$.

Turning on discrete torsion in the string path integral simply means inserting an overall minus sign in the twisted sector associated with each of these three elements when projecting onto group invariant states. (We give the explicit form of the discrete torsion in equation (17) below.)

Here (using the notation of equations (1) – (3)), if we consider the twisted sector associated with α , we have to project onto states invariant under the group generated by β and γ . In this case, we are free to insert an overall minus sign in the action of β , γ and $\beta\gamma$ on the α twisted sector. This phase is discrete torsion. There exist analogous choices in the remaining two twisted sectors. In the example we are considering here, it is straightforward to check

that in the twisted sectors corresponding to the first two generators, all three choices for the torsion give the same contribution to the Betti numbers as the case without discrete torsion. For the eight three-tori associated with the third generator in this example, turning on discrete torsion means that states which were odd under the Z_2 in equation (8) in the case without the torsion are now even and vice versa. Geometrically this corresponds to the fact that these singularities admit two topologically distinct resolutions [11, 15]. For us it means that the orbifold with discrete torsion has Betti numbers $b_2 = 8$ and $b_3 = 47$.

Therefore if we could prove that the transformation T_{235} as well as converting IIA to IIB also turns on discrete torsion, then we would have proved in this case our extended PT conjecture.

If we begin with Type IIA theory on the orbifold without any discrete torsion then we have Type IIA on the Joyce manifold with $b_2 = 16$ and $b_3 = 39$. After making the transformation T_{235} we would then end up with Type IIB on the Joyce manifold with $b_2 = 8$ and $b_3 = 47$. However, as shown in [15], in string theory one can smoothly interpolate between any of the manifolds in this family (ie any value for l) by turning on marginal perturbations. Thus at the level of conformal field theory the backgrounds on the IIA and IIB sides are equivalent. Thus if we could prove that the IIA theory without discrete torsion is equivalent to the IIB theory with discrete torsion, then we have essentially proven the PT conjecture.

We will now complete the proof of the PT conjecture along the lines of the computation in [19].

4.1 Genus g Transformation of Path Integral.

A genus g Riemann surface, Σ^g , has 2^{2g} spin structures. If the number of fermion zero modes of a given chirality is even or odd, then a given spin structure, α is said to be even or odd. It is a well known fact that under an $R \rightarrow 1/R$ transformations on circles in, say, the x_i directions, the left-moving fermions which partner the string coordinates in those directions pick up a minus sign. In fact this is one of the fundamental reasons why the Type IIA and IIB theories are related by T -duality. The non-zero modes of these fermions are naturally paired, so these modes will not contribute any change in sign to the measure of the path integral. Thus, under such T -duality transformations, the transformation of the measure of the path integral is

determined by the number of fermion zero-modes in the transformed directions being even or odd. Thus, for spin structure α , the measure, μ , of the genus g path integral of a toroidal target space transforms as [19]

$$\mu_{g,\alpha} \rightarrow (-)^{\sigma_\alpha} \mu_{g,\alpha} \quad (15)$$

where σ_α is 0 or 1 if the spin structure α is even or odd. The above formula asserts that the IIA/IIB theory on a torus transforms into the IIB/IIA theory on the T -dual torus, under an odd number of $R \rightarrow 1/R$ transformations. Before discussing the orbifold case, we will briefly define the spin structure parity, σ_α for a genus g Riemann surface, Σ^g . This was discussed in [19].

On Σ^g , we can choose a canonical basis of 1-cycles, (a_i, b_i) , for $i = 1, \dots, g$. This basis defines a canonical spin structure. Thus, we can specify a spin structure $\alpha = (\theta_i, \phi_j) \equiv (\Theta, \Phi)$ by the Z_2 valued quantities Θ and Φ ((Θ, Φ) are g -vectors with components (θ_i, ϕ_i)). In the spin structure α , fermions have an extra minus sign in their (a_i, b_i) boundary conditions, relative to the canonical spin structure, such that the components of Θ or Φ are one [19]. The parity of the genus g spin structure is

$$\sigma_\alpha \equiv \sigma(\Theta, \Phi) = \Theta \cdot \Phi \mod 2 \quad (16)$$

Now we discuss the modification to the above transformation for the toroidal orbifolds we have been discussing in this paper. Our orbifolds are of the form T^7/Γ , with Γ the Z_2^3 group generated by α, β, γ of equations (1) – (3). We must therefore consider the Γ twists around the a_i and b_i directions. In the (a_i, b_i) directions, a general Γ twist will be of the form $(\alpha^{r_i} \beta^{s_i} \gamma^{t_i}, \alpha^{u_i} \beta^{v_i} \gamma^{w_i})$, or $(\alpha^R \beta^S \gamma^T, \alpha^U \beta^V \gamma^W)$ in g -vector notation.

The discrete torsion for this theory is given by

$$\epsilon = (-1)^{R \cdot V - S \cdot U + R \cdot W - T \cdot U + S \cdot W - T \cdot V} \quad (17)$$

This formula is the generalisation to the Z_2^3 case of the Z_2^2 case given in [19]. A simple way to see this is that the above discrete torsion should reduce to that considered in [19], when any one of the three Z_2 generators acts trivially; ie consider the twists when any single pair of g -vectors, $(R, U), (S, V), (T, W)$ is zero. When this is the case, the twist reduces to a $Z_2 \times Z_2$ and therefore the formula should reproduce that of [19]. Thus the above formula is the unique generalisation of the one given in [19]. In fact

the formula above is just three copies of that in [19], one for each $Z_2 \times Z_2$ subgroup of Γ .

Having calculated the discrete torsion phase factor for the Joyce orbifolds which have been our interest, our aim is to show that, under T_{235} , the path integral measure at genus g , with spin structure α transforms as

$$\mu_{g,\alpha} \rightarrow (-)^{\sigma_\alpha} \cdot \epsilon \cdot \mu_{g,\alpha} \quad (18)$$

with σ and ϵ as defined in equations (16) and (17). If this formula is true, then the IIA/IIB theory without (or with) discrete torsion transforms under T_{235} into the IIB/IIA theory with (or without) discrete torsion. The entire transformation would be a consequence of the fact that left-moving zero modes in the T -dualised directions change sign. We can view the transformation as follows. The first factor converts IIA/IIB into IIB/IIA, while the second turns discrete torsion on (or off), if the undualised theory had discrete torsion turned off (or on) [19].

Since we are inverting radii in the $(2, 3, 5)$ directions, and it is the fermions (which superpartner the string coordinates in these directions) which change sign in the toroidal compactification, we need to identify the “twisting” of a toroidal spin structure in these directions, say $\alpha = (\Theta, \Phi)$ when twisted by the orbifold group. By inspecting the action of α, β, γ on these coordinates, and with the discrete torsion defined above, it is easy to see that the spin structure, (Θ, Φ) is shifted in these directions to the following:

$$x_2 : (\Theta + R + S, \Phi + U + V) \quad (19)$$

$$x_3 : (\Theta + R + T, \Phi + U + W) \quad (20)$$

$$x_5 : (\Theta + S + T, \Phi + V + W) \quad (21)$$

In the orbifold theory, the path integral transforms as follows:

$$\mu_{g,\alpha} \rightarrow (-)^{[\sigma(\Theta+R+S, \Phi+U+V) + \sigma(\Theta+R+T, \Phi+U+W) + \sigma(\Theta+S+T, \Phi+V+W)]} \cdot \mu_{g,\alpha} \quad (22)$$

Using (16) and (17), or Theorem 2 of [22] (which is also stated in [19]) which states:

$$\sigma(\Theta+R+U, \Phi+S+T) = \sigma(\Theta, \Phi) + \sigma(\Theta+R, \Phi+S) + \sigma(\Theta+U, \Phi+T) + (R.T - S.U) \quad (23)$$

it is straightforward to verify that the above transformation is precisely the required transformation (18). This completes the proof that the IIA/IIB theory on the T^7/Γ orbifold without (or with) discrete torsion is equivalent to the IIB/IIA theory on the $T^{7'}/\Gamma$ orbifold with (or without) discrete torsion, where the prime denotes that the two 7-tori are related by T_{235} . We now discuss the implications of this proof for the SV conjecture.

4.2 Relation to SV Conjecture.

Up to this point we have had very little to say about the SV generalised mirror conjecture [15]. We will now show that our proof of equation (18) and our discussions above in fact tell us that the SV conjecture and the PT conjecture are in fact the same conjecture for the cases to which we have restricted ourselves. In fact, being much more general, the SV conjecture turns out to encompass the PT conjecture, with the latter a special case of the former. By establishing this fact momentarily, our proof of (18) also yields concrete examples of cases in which the SV conjecture applies.

The reason that the conjectures coincide for the compactifications which have interested us here is the following. Under the duality transformation, T_{235} which interchanges the IIA and IIB theories, the operator, $(-)^{F_l}$, changes sign. This means that $(-)^F$ also changes sign. This is precisely the kind of situation to which the SV generalised mirror conjecture applies. As we have shown that the two theories are exactly equivalent under T_{235} , and also that this proves the PT conjecture for these cases, the SV and PT conjectures for IIA/IIB compactification on manifolds of G_2 holonomy are in fact the same.

5 Discussion.

We have discussed in this paper perhaps the simplest class of Type II vacua in three dimensions which have $N = 2$ supersymmetry. We hope we have convinced the reader that there is a rich structure in this class of vacua which deserves to be further explored. One interesting avenue for this is the following. The strong coupling limit of a $d = 3$, $N = 2$ Type IIA theory, obtained by compactifying on some “ G_2 ” background, J , is a $d = 4$, $N = 1$ vacuum obtained by compactifying M -theory on J . It would certainly be interesting to study which kind of physics this will lead to. In fact since

manifolds of G_2 holonomy contain only supersymmetric 3-cycles and 4-cycles, M -theory on J can have supersymmetric membrane instantons which come from wrapping membranes on such 3-cycles. It is plausible that these could generate a non-perturbative superpotential via a mechanism discussed in [28]. The magnetic duals of these objects are strings which come from wrapping fivebranes on 4-cycles. Moreover, when such 4-cycles vanish, one should get an interesting tensionless string appearing in four dimensions. Tensionless strings in four dimensions have been discussed recently in [29].

For this same simple class of vacua, we have also given a proof of the PT and SV “mirror” conjectures using T -duality. This, we believe puts these “mirror” conjectures on a much firmer footing. We hope that this convinces the reader that the geometry and physics of more general “ G_2 ” vacua is possibly as interesting as the Type II Calabi-Yau “mirror vacua”.

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